Cauchy Confers with Weierstrass

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Abstract. We point out two limitations of using the Cauchy Residue Theorem to evaluate a definite integral of a real rational function of $\cos\theta$ and $\sin\theta$. We also exhibit a definite integral with the limitations. Finally, we overcome the limitations by using the Weierstrass substitution.

Introduction

Let *R* be a rational function of two real variables. Consider a real integral

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\,d\theta .$$

By using the substitution $z = e^{i\theta}$, where $i=\sqrt{-1}$ and θ is a real number, it can be shown that

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\,d\theta = -i\oint_{|z|=1} \tfrac{1}{z} R\left(\tfrac{z^2+1}{2z}, \tfrac{z^2-1}{2iz}\right)dz . \tag{1}$$

For details, see, for example, Palka [4], p. 329 or Ahlfors [1], p. 154 or Crowder and McCuskey [2], p. 258. By the Cauchy Residue Theorem, we have

$$-i\oint_{|z|=1} \tfrac{1}{z} R\left(\tfrac{z^2+1}{2z}, \tfrac{z^2-1}{2iz}\right)dz = (-i)(2\pi i)\sum_{k=1}^{n} \operatorname{Re} s(f(z); z_k) = 2\pi \sum_{k=1}^{n} \operatorname{Re} s(f(z); z_k),$$

where

$$f(z) = \tfrac{1}{z} R\left(\tfrac{z^2+1}{2z}, \tfrac{z^2-1}{2iz}\right),$$

$z_1, \ldots, z_n$ are the poles of *f* in the open unit disk $D = \{z \in C : |z| < 1\}$, and Res $(f(z); z_k)$ denotes the residue of *f* at the pole z_k. Hence, (1) becomes

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\,d\theta = 2\pi \sum_{k=1}^{n} \operatorname{Re} s(f(z); z_k). \tag{2}$$

Limitations of Formula (2)

In this section, we point out two cases for which the formula (2) does not apply and we suggest a different method of attack.

Case # 1: At least one pole of f lies on the unit circle {z∈C:|z|=1}.
Case # 2: The function R is not an even function of θ and one wishes to evaluate

$$\int_0^{\alpha} R(\cos\theta, \sin\theta)d\theta, \text{ where } 0<\alpha<2\pi.$$

We give an example of such a function *f*. Let

$$R(\cos\theta, \sin\theta) = \frac{25\cos\theta}{3\sin\theta + 4\cos\theta}.$$

Then

$$R\left(\frac{z^2+1}{2z}, \frac{z^2-1}{2iz}\right) = \frac{25\left(\frac{z^2+1}{2z}\right)}{3\left(\frac{z^2-1}{2iz}\right)+4\left(\frac{z^2+1}{2z}\right)} = \frac{25i(z^2+1)}{(3+4i)z^2-3+4i},$$

so that

$$f(z) = \frac{1}{z} R\left(\frac{z^2+1}{2z}, \frac{z^2-1}{2iz}\right) = \frac{25i(z^2+1)}{z((3+4i)z^2-3+4i)}.$$

The poles of f are found by solving $z((3+4i)z^2 - 3 + 4i) = 0$ for z. We get $z = 0$ or $(3+4i)z^2 - 3 + 4i = 0$. Using the Quadratic Formula, the solutions to $(3+4i)z^2 - 3 + 4i = 0$ are $z = \frac{3}{5} - \frac{4}{5}i$ and $z = -\frac{3}{5} + \frac{4}{5}i$. Hence, the poles of f are z = 0, $z = \frac{3}{5} - \frac{4}{5}i$, and $z = -\frac{3}{5} + \frac{4}{5}i$. Since $|\frac{3}{5} - \frac{4}{5}i| = |-\frac{3}{5} + \frac{4}{5}i| = 1$, we conclude that the poles $z = \frac{3}{5} - \frac{4}{5}i$ and $z = -\frac{3}{5} + \frac{4}{5}i$ of f lie on the unit circle {z∈C:|z|=1}. Since the rational function R is not even, we see that the integral

$$\int_0^{\pi/2} \frac{25\cos\theta}{3\sin\theta + 4\cos\theta}\, d\theta \tag{3}$$

is an example of a real integral which satisfies Cases 1 and 2 and so we cannot use formula (2) to evaluate it.

Remedy: The Weierstrass Substitution

In this section, we use the well-known Weierstrass substitution to evaluate (3). Let

$$t = \tan\left(\frac{\theta}{2}\right), \text{ for } \theta\in(-\pi,\pi).$$

Then

$$\cos\left(\frac{\theta}{2}\right) = \frac{1}{\sqrt{t^2+1}} \tag{4}$$

and

$$\sin\left(\frac{\theta}{2}\right) = \frac{t}{\sqrt{t^2+1}}. \tag{5}$$

Replacing A by $\frac{\theta}{2}$ in the following double-angle-formula

$$\sin(2A) = 2\sin A\cos A,$$

and using (4) and (5) we have

$$\sin\theta = \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = \frac{2t}{t^2+1}.$$

Similarly, replacing A by $\frac{\theta}{2}$ in the following double-angle-formula

$$\cos(2A) = \cos^2 A - \sin^2 A,$$

and using (4) and (5) we have

$$\cos\theta = \cos^2(\theta/2) - \sin^2(\theta/2) = \frac{1-t^2}{t^2+1}.$$

Also, from

$$t = \tan\left(\frac{\theta}{2}\right),$$

we have

$$\theta = 2\tan^{-1} t, \text{ for } \theta \in (-\pi, \pi),$$

from which we obtain

$$d\theta = \frac{2}{t^2+1}\,dt.$$

Hence

$$\int R(\cos\theta, \sin\theta)\,d\theta = \int R\left(\frac{1-t^2}{t^2+1}, \frac{2t}{t^2+1}\right)\cdot\frac{2}{t^2+1}\,dt. \qquad (6)$$

For details on the Weierstrass substitution, see, for example, Holder [3], p. 339. We are now ready to employ the Weierstrass substitution to evaluate

$$\int_0^{\frac{\pi}{2}} \frac{25\cos\theta}{3\sin\theta+4\cos\theta}\,d\theta.$$

First, we change the limits of integration using $t = \tan(\theta/2)$. When $\theta = 0$, then $t = \tan(0/2) = \tan(0) = 0$. When $\theta = \frac{\pi}{2}$, then $t = \tan(\pi/4) = 1$. Now, using (6), we have

$$\int_0^{\frac{\pi}{2}} \frac{25\cos\theta}{3\sin\theta+4\cos\theta}\,d\theta = \int_0^1 \frac{\frac{25(1-t^2)}{t^2+1}}{3\left(\frac{2t}{t^2+1}\right)+4\left(\frac{1-t^2}{t^2+1}\right)}\cdot\frac{2}{t^2+1}\,dt = \int_0^1 \frac{25t^2-25}{(2t+1)(t-2)(t^2+1)}\,dt.$$

By partial fraction decomposition, we have

$$\frac{25t^2-25}{(2t+1)(t-2)(t^2+1)} = \frac{6}{2t+1} + \frac{3}{t-2} - \frac{6t-8}{t^2+1} = \frac{6}{2t+1} + \frac{3}{t-2} - \frac{6t}{t^2+1} + \frac{8}{t^2+1}.$$

Therefore

$$\int_0^{\frac{\pi}{2}} \frac{25\cos\theta}{3\sin\theta+4\cos\theta}d\theta = \int_0^1 \frac{25t^2-25}{(2t+1)(t-2)(t^2+1)}dt = \int_0^1 \left(\frac{6}{2t+1} + \frac{3}{t-2} - \frac{6t}{t^2+1} + \frac{8}{t^2+1}\right)dt$$

$$= \Big[3\ln|2t+1| + 3\ln|t-2| - 3\ln(t^2+1) + 8\tan^{-1}t\Big]_0^1 = 3\ln\left(\tfrac{3}{4}\right) + 2\pi.$$

Conclusion

If R is a rational function of two real variables, then it is labor-saving if we are able to evaluate real integrals of the form

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)d\theta \qquad (7)$$

using formula (2). However, as we have seen, there exist some stubborn real integrals of the form $\int_0^{\alpha} R(\cos\theta, \sin\theta)d\theta$ for which formula (2) is of no use. The good news is that the Weierstrass substitution whips all rational integrals of the form $\int_0^{\alpha} R(\cos\theta, \sin\theta)d\theta$ into submission!

Biographical Sketch

Lloyd Moyo received his B.Ed (Science) in 1992 from the University of Malawi in southern Africa. He received his M.Sc. in Mathematics from the University of Sussex, U.K. in 1996 and his Ph.D. in Mathematics from New Mexico State University in 2006. He joined Henderson State University in Fall 2012. He is a member of the American Mathematical Society and the Mathematical Association of America.

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Conceptions of Innocence and Experience in Blake's "The Book of Thel" and Wordsworth's "Intimations Ode"

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Abstract

Both William Blake's "The Book of Thel" and William Wordsworth's "Ode: Intimations of Immortality from Recollections of Early Childhood" explore the effects of leaving a perfect, enlightened world and entering the flawed domain of man while suggesting the pre-existence of the human soul. Wordsworth's ode envisions the world as an imperfect, but worthwhile environment while Blake's poem focuses on the dangers and pains that accompany it. The authors paint differing pictures of this world of experience by personifying nature, employing vivid imagery of the soul before and after its descent, and by using significant symbols in their respective epigraphs to reflect their overall views. These devices work together to provide the reader with a choice of how to handle this knowledge of the once divine, but now unkind, universe.

Essay

It is the "obstinate questionings / of sense and outward things" that constitute the bulk of human life in both William Blake's "The Book of Thel" and William Wordsworth's "Ode: Intimations of Immortality from Recollections of Early Childhood" (Wordsworth 9.141-42). Questioning the world in which they live, both authors explore the effects of leaving a perfect, enlightened world and entering the flawed domain of man while suggesting the pre-existence of the human soul. Wordsworth's ode envisions the world as an imperfect, but worthwhile environment while Blake's poem focuses on the dangers and pains that accompany it. The authors paint differing pictures of this world of experience by personifying nature, employing vivid imagery of the soul before and after its descent, and by using significant symbols in their respective epigraphs to reflect their overall views. These devices work together to provide the reader with a choice of how to handle this knowledge of the once divine, but now unkind, universe.

A Romantic idea of nature as something not only alive, but also completely sentient is explored quite literally in Blake's "The Book of Thel". He uses personification to form a dialogue between the innocent yet unsatisfied Thel and the other inhabitants of the Vales of Har. Described as "small" and "lowly," the Lilly answers Thel's cries of dissatisfaction concerning her lack of purpose in life (Blake 1.1.17). The Lilly wants to help Thel, as do all of Blake's personified natural elements, crawl out of her melancholia. The Lilly's speech