Great (But Lesser Known) Theorems

Fred Worth, Ph.D.
Professor of Mathematics

Abstract - Mathematics is filled with wonderful theorems. Many of these theorems are well known, such as Fermat's Last Theorem, the Pythagorean Theorem, and the First and Second Fundamental Theorems of Calculus. There are, however, some wonderful theorems that are not nearly so well known. The purpose of this paper is to present three of those results.

Sturm's Theorem

I encountered this theorem in Nathan Jacobson's wonderful book Basic Algebra I (2e). It is credited to Jacques Charles François Sturm (September 29, 1803 - December 15, 1855).

First, some terminology.

The **Standard Sequence** for a polynomial f(x) is
$f_0(x) = f(x)$
$f_1(x) = f'(x)$
$f_0(x) = q_1(x)\, f_1(x) - f_2(x)$, deg $f_2 <$ deg f_1
$f_1(x) = q_2(x)\, f_2(x) - f_3(x)$, deg $f_3 <$ deg f_2
. . .
$f_{i-1}(x) = q_i(x)\, f_i(x) - f_{i+1}(x)$, deg $f_{i+1} <$ deg f_i
. . .
$f_{s-1}(x) = q_s(x)\, f_s(x)$, that is, $f_{s+1}(x) = 0$

Note that, for $n > 1$, f_n is the negative of the remainder from the division algorithm.

Sturm's Theorem - Let f(x) be a polynomial of positive degree with real coefficients and let $\{f_0(x) = f(x), f_1(x) = f'(x), f_2(x), \ldots, f_s(x)\}$ be the standard sequence for f(x). Assume [a, b] is an interval such that $f(a) \neq 0 \neq f(b)$. Then the number of distinct roots of f(x) in (a, b) is $V_a - V_b$ where V_c denotes the number of variations in sign of $\{f_0(c), f_1(c), \ldots, f_s(c)\}$. [Note - drop any 0's from the sequence.]

Here is an example, using Maple.

```
> f0:=x^3-7*x+6; f1:=diff(f0,x); f2:=-rem(f0,f1,x); f3:=-rem(f1,f2,x); f4:=-rem(f2,f3,x);
```

$$f0 := x^3 - 7x + 6$$

$$f1 := 3x^2 - 7$$

$$f2 := -6 + \frac{14}{3}x$$

$$f3 := \frac{100}{49}$$

$$f4 := 0$$

```
> [subs(x=0,f0),subs(x=0,f1),subs(x=0,f2),subs(x=0,f3)];
```

$$\left[6, -7, -6, \frac{100}{49}\right]$$

```
> [subs(x=1.5,f0),subs(x=1.5,f1),subs(x=1.5,f2),subs(x=1.5,f3)];
```

$$\left[-1.125, -0.25, 1.000000000, \frac{100}{49}\right]$$

```
> [subs(x=3,f0),subs(x=3,f1),subs(x=3,f2),subs(x=3,f3)];
```

$$\left[12, 20, 8, \frac{100}{49}\right]$$

```
> fsolve(f0,x);
```

$$-3., 1., 2.$$

Note that there are two sign changes for x = 0, one for x = 1.5 and none for x = 3. Thus, there are 2-1 = 1 roots between 0 and 1.5, 2-0 = 2 between 0 and 3, and 1-0 = 1 between 1.5 and 3.

Now let us consider another example.

```
> f0:=x^5-7.8*x^4+22.36*x^3-29.16*x^2+16.96*x-3.36;
  f1:=diff(f0,x); f2:=-rem(f0,f1,x); f3:=-rem(f1,f2,x);
  f4:=-rem(f2,f3,x);f5:=-rem(f3,f4,x); f6:=-rem(f4,f5,x);
```

$$f0 := x^5 - 7.8x^4 - 3.36 + 22.36x^3 + 16.96x - 29.16x^2$$

$$f1 := 5x^4 - 31.2x^3 + 67.08x^2 + 16.96 - 58.32x$$

$$f2 := -1.931520000 + 4.627840000x - 3.432960000x^2 + 0.7904000000x^3$$

$$f3 := 6.214820764 - 9.424511138x + 3.384746513x^2$$

$$f4 := -0.3308898615 + 0.2542821669x$$

$$f5 := 0.3176096024$$

$$f6 := 0$$

```
> [subs(x=0,f0),subs(x=0,f1),subs(x=0,f2),subs(x=0,f3),subs(x=0
,f4),subs(x=0,f5)];
        [-3.36, 16.96, -1.931520000, 6.214820764, -0.3308898615, 0.3176096024]
> [subs(x=1.5,f0),subs(x=1.5,f1),subs(x=1.5,f2),subs(x=1.5,f3),
subs(x=1.5,f4),subs(x=1.5,f5)];
        [0.04125, 0.4225, -0.046320000, -0.306266292, 0.0505333889, 0.3176096024]
> [subs(x=3,f0),subs(x=3,f1),subs(x=3,f2),subs(x=3,f3),subs(x=3
,f4),subs(x=3,f5)];
        [0., 8.32, 2.39616000, 8.40400597, 0.4319566392, 0.3176096024]
> solve(f0,x);
        1., 2., 3., 0.4000000000, 1.400000000
>
```

Note that there are five sign changes for x = 0, two for x = 1.5 and none for x = 3. Thus, there are 5-2 = 3 roots between 0 and 1.5, 5-0 = 5 roots between 0 and 3, and 2-0 = 2 between 1.5 and 3.

Sarkovskii's Theorem

The next result is something I first encountered in Robert L. Devaney's great book An Introduction to Chaotic Dynamical Systems (2e)

Terminology

$f^1 = f$

$f^n = f \circ f^{n-1}$

The point, x, is a fixed point of f if f(x) = x.

The point, x, is a periodic point of period n for f if x is a fixed point of f^n and is not a fixed point for f^k for any k < n.

We now define an ordering on the positive integers.

Sarkovskii's Ordering

$3 \rhd 5 \rhd 7 \rhd \ldots \rhd 2\cdot 3 \rhd 2\cdot 5 \rhd \ldots \rhd 2^2\cdot 3 \rhd 2^2\cdot 5 \rhd \ldots \rhd 2^3\cdot 3 \rhd 2^3\cdot 5 \rhd \ldots \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1.$

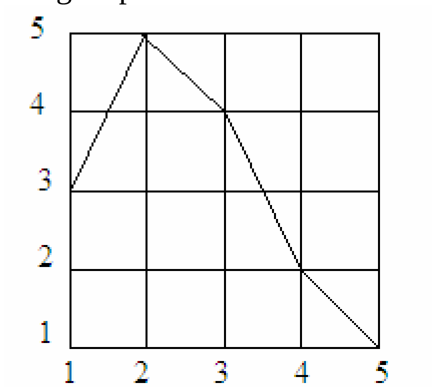
Theorem - Suppose f: $\mathbf{R} \to \mathbf{R}$ is continuous. Suppose f has a periodic point of period k. If $k \rhd n$ in the Sarkovskii ordering, then f also has a periodic point of period n.

Sarkovskii's Theorem first appeared in the Ukrainian Maths Journal in 1964 (Vol 16, pp 61-71).

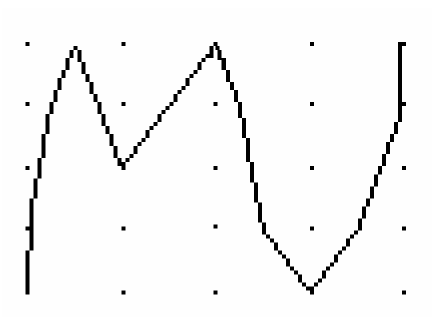
In December 1975, Tien-Yien Li and James A. Yorke published a paper in the *American Mathematical Monthly* (**82**, 985-992.) entitled "Period Three Implies Chaos." This paper proved a similar result on intervals. They didn't mention Sarkovskii. Due to the poor communication between Eastern Europe and the United States mathematical communities, it is possible they were unaware of his result.

A converse of Sarkovskii's is also true. There are maps which have periodic points of period k and no "higher" period according to the Sarkovskii ordering.

For example, consider the following map.



Let f(x) = 3 for x < 1 and f(x) = 1 for x > 5. $f^5(1) = f^4(3) = f^3(4) = f^2(2) = f(5) = 1$. Thus, 1, 2, 3, 4 and 5 are all period 5 points. Note that $f^3[1, 2] = [2, 5]$, $f^3[2, 3] = [3, 5]$ and $f^3[4, 5] = [1, 4]$ so the only possible period 3 points are in [3, 4] since $f^3[3, 4] = [1, 5]$. We see f[3, 4] above. It contains a fixed point, x = 10/3. But, note from the graph below of f^3 that f^3 is monotonically decreasing on [3, 4] so the only fixed point of f^3 is 10/3, the fixed point of f.



No point outside of [1, 5] will be periodic since all iterations of them will be in [1, 5]. Thus we have a period 5 point but no period 3 point.

Similar functions can be constructed having period 2n+1 points but no 2n-1 points for n > 2.

For the even periods, the construction is a little messier and I'll refer you to Devaney's book.

Solution by Radicals of 3rd and 4th Degree Polynomials

College algebra students are familiar with the quadratic formula, a method for solving second degree polynomial equations by radicals. I first encountered the analogous formulas for third and fourth degree polynomials in the fascinating book An Introduction to the Theory of Groups by Joseph Rotman.

Cubic Formula

Consider $x^3 + Ax^2 + Bx + C$.

Let X = x + A/3. Then

$$x^3 + Ax^2 + Bx + C = \left(X - \frac{A}{3}\right)^3 + A\left(X - \frac{A}{3}\right)^2 + B\left(X - \frac{A}{3}\right) + C$$

$$= X^3 - AX^2 + \frac{XA^2}{3} - \frac{A^3}{27} + AX^2 - \frac{2XA^2}{3} + \frac{A^3}{9} + BX - \frac{AB}{3} + C$$

$$= X^3 + X^2(-A + A) + X\left(\frac{A^2}{3} - \frac{2A^2}{3} + B\right) + \left(\frac{-A^3}{27} + \frac{A^3}{9} - \frac{AB}{3} + C\right)$$

$$= X^3 + 0\,X^2 + Q\,X + R$$

$$= X^3 + Q\,X + R$$

where Q = $\frac{A^2}{3} - \frac{2A^2}{3} + B$ and R = $\frac{-A^3}{27} + \frac{A^3}{9} - \frac{AB}{3} + C$.

Suppose $a^3 + Qa + R = 0$.

Write a = b + c where b and c are to be determined.

Then $a^3 = (b + c)^3 = b^3 + c^3 + 3(cb^2 + bc^2) = b^3 + c^3 + 3abc$.

Then $b^3 + c^3 + (3bc + Q)a + R = 0$. (*)

Also suppose bc = -Q/3. Then the middle term in (*) vanishes.

Therefore (*) becomes $b^3 + c^3 = -R$ and $b^3c^3 = -Q^3/27$. Thus, from substitution we have

$$b^3 - \frac{Q^3}{27b^3} = -R,$$

or

$$\left(b^3\right)^2 + R\,b^3 - \frac{Q^3}{27} = 0.$$

By the quadratic formula we have

$$b^3 = \frac{-R \pm \sqrt{R^2 + \frac{4Q^3}{27}}}{2}$$

and, since $b^3 + c^3 = -R$,

$$c^3 = \frac{-R \mp \sqrt{R^2 + \frac{4Q^3}{27}}}{2}.$$

Thus

$$a = b + c = \left(\frac{-R}{2} + \sqrt{\frac{R^2}{4} + \frac{Q^3}{27}} \right)^{1/3} + \left(\frac{-R}{2} - \sqrt{\frac{R^2}{4} + \frac{Q^3}{27}} \right)^{1/3}$$

where the cube roots are chosen so the product bc = -Q/3. This is a root of the altered cubic equation so x = a - A/3. Once that root is divided out you have a quadratic which can be solved easily.

$x^3 + Ax^2 + Bx + C$.

x = a - A/3

$$= \left(\frac{-R}{2} + \sqrt{\frac{R^2}{4} + \frac{Q^3}{27}} \right)^{1/3} + \left(\frac{-R}{2} - \sqrt{\frac{R^2}{4} + \frac{Q^3}{27}} \right)^{1/3} - \frac{A}{3}$$

$$= \left(\frac{-\left(\frac{2A^3}{27} - \frac{AB}{3} + C \right)}{2} + \sqrt{\frac{\left(\frac{2A^3}{27} - \frac{AB}{3} + C \right)^2}{4} + \frac{\left(B - \frac{A^2}{3} \right)^3}{27}} \right)^{1/3}$$

$$+ \left(\frac{-\left(\frac{2A^3}{27} - \frac{AB}{3} + C \right)}{2} - \sqrt{\frac{\left(\frac{2A^3}{27} - \frac{AB}{3} + C \right)^2}{4} + \frac{\left(B - \frac{A^2}{3} \right)^3}{27}} \right)^{1/3} - \frac{A}{3}$$

$$= \left(-\frac{1}{27}A^3 + \frac{1}{6}BA - \frac{1}{2}C + \frac{1}{18}\sqrt{81\left(\frac{2}{27}A^3 - \frac{1}{3}BA + C\right)^2 + 12\left(B - \frac{1}{3}A^2\right)^3} \right)^{(1/3)}$$
$$+ \left(-\frac{1}{27}A^3 + \frac{1}{6}BA - \frac{1}{2}C - \frac{1}{18}\sqrt{81\left(\frac{2}{27}A^3 - \frac{1}{3}BA + C\right)^2 + 12\left(B - \frac{1}{3}A^2\right)^3} \right)^{(1/3)} - \frac{1}{3}A$$

$x^3 + Ax^2 + Bx + C = 0$ has solution

$$a = \left(-\frac{1}{27}A^3 + \frac{1}{6}BA - \frac{1}{2}C + \frac{1}{18}\sqrt{81\left(\frac{2}{27}A^3 - \frac{1}{3}BA + C\right)^2 + 12\left(B - \frac{1}{3}A^2\right)^3} \right)^{(1/3)}$$
$$+ \left(-\frac{1}{27}A^3 + \frac{1}{6}BA - \frac{1}{2}C - \frac{1}{18}\sqrt{81\left(\frac{2}{27}A^3 - \frac{1}{3}BA + C\right)^2 + 12\left(B - \frac{1}{3}A^2\right)^3} \right)^{(1/3)} - \frac{1}{3}A$$

and $x^3 + Ax^2 + Bx + C = (x - a)(x^2 + bx + c)$.

Quartic Formula

For the quartic $x^4 + Ax^3 + Bx^2 + Cx + D$, let $x = X - A/4$. This gives

$$x^4 + Ax^3 + Bx^2 + Cx + D$$
$$= \left(X - \frac{A}{4}\right)^4 + A\left(X - \frac{A}{4}\right)^3 + B\left(X - \frac{A}{4}\right)^2 + C\left(X - \frac{A}{4}\right) + D$$
$$= X^4 - AX^3 + \frac{3A^2X^2}{8} - \frac{XA^3}{16} + \frac{A^4}{256} + AX^3 - \frac{3X^2A^2}{4}$$
$$+ \frac{3XA^3}{16} - \frac{A^4}{64} + BX^2 - \frac{BAX}{2} + \frac{BA^2}{16} + CX - \frac{AC}{4} + D$$
$$= X^4 + X^3(-A + A) + X^2\left(\frac{3A^2}{8} - \frac{3A^2}{4} + B\right)$$
$$+ X\left(\frac{-A^3}{16} + \frac{3A^3}{16} - \frac{BA}{2} + C\right) + \left(\frac{A^4}{256} - \frac{A^4}{64} + \frac{BA^2}{16} - \frac{AC}{4} + D\right)$$
$$= X^4 + QX^2 + RX + S$$

where

$$Q = \frac{3\,A^2}{8} - \frac{3\,A^2}{4} + B,\; R = \frac{-A^3}{16} + \frac{3\,A^3}{16} - \frac{B\,A}{2} + C\,,\; S = \frac{A^4}{256} - \frac{A^4}{64} + \frac{B\,A^2}{16} - \frac{A\,C}{4} + D.$$

$X^4 + QX^2 + RX + S = (X^2 + KX + L)(X^2 - KX + M)$ where finding K, L and M reduces us to two quadratics.

Expanding the right hand side and equating coefficients of like powers of x gives $L + M - K^2 = Q$, $K(M - L) = R$ and $LM = S$.

Thus, from the first two of these we get
$2M = K^2 + Q + R/K$ and $2L = K^2 + Q - R/K$. (**)
Substituting these into the third equation gives $(K^3 + QK + R)(K^3 + QK - R) = 4SK^2$ which becomes $K^6 + 2QK^4 + (Q^2 - 4S)K^2 - R^2 = 0$ which is a cubic in K^2 and can thus be solved using the technique given above.

We then can solve for L and M in (**) giving two quadratics which we can solve using the quadratic formula.

One solution is:

$$x1 :=$$
$$-\frac{1}{12}\,\mathrm{sqrt}(6\,(8\,B^3 - 36\,A\,B\,C + 108\,d\,A^2 - 288\,d\,B + 108\,C^2 + 12\,\mathrm{sqrt}(-54\,A^3\,B\,C\,d + 240\,A\,B^2\,C\,d + 12\,B^3\,d\,A^2 - 3\,A^2\,B^2\,C^2$$
$$-54\,A\,B\,C^3 - 432\,d^2\,A^2\,B + 18\,d\,A^2\,C^2 - 432\,d\,B\,C^2 - 48\,B^4\,d + 12\,B^3\,C^2 + 81\,d^2\,A^4 + 384\,d^2\,B^2 + 12\,C^3\,A^3 + 81\,C^4$$
$$-768\,d^3 + 576\,C\,A\,d^2))\hat{}(1/3) + 6\,(8\,B^3 - 36\,A\,B\,C + 108\,d\,A^2 - 288\,d\,B + 108\,C^2 - 4\,\mathrm{sqrt}(59049 - 4\,B^6 + 36\,B^4\,A\,C$$
$$-144\,B^4\,d - 108\,A^2\,B^2\,C^2 + 864\,A\,B^2\,C\,d - 1728\,d^2\,B^2 + 108\,C^3\,A^3 - 1296\,d\,A^2\,C^2 + 5184\,C\,A\,d^2 - 6912\,d^3))\hat{}(1/3)$$
$$+9\,A^2 - 24\,B) + \frac{1}{12}\,\mathrm{sqrt}\Big(6\,(8\,B^3 - 36\,A\,B\,C + 108\,d\,A^2 - 288\,d\,B + 108\,C^2 + 12\,\mathrm{sqrt}(-54\,A^3\,B\,C\,d + 240\,A\,B^2\,C\,d$$
$$+12\,B^3\,d\,A^2 - 3\,A^2\,B^2\,C^2 - 54\,A\,B\,C^3 - 432\,d^2\,A^2\,B + 18\,d\,A^2\,C^2 - 432\,d\,B\,C^2 - 48\,B^4\,d + 12\,B^3\,C^2 + 81\,d^2\,A^4$$
$$+384\,d^2\,B^2 + 12\,C^3\,A^3 + 81\,C^4 - 768\,d^3 + 576\,C\,A\,d^2))\hat{}(1/3) + 6\,(8\,B^3 - 36\,A\,B\,C + 108\,d\,A^2 - 288\,d\,B + 108\,C^2 - 4\,\mathrm{sqrt}($$
$$59049 - 4\,B^6 + 36\,B^4\,A\,C - 144\,B^4\,d - 108\,A^2\,B^2\,C^2 + 864\,A\,B^2\,C\,d - 1728\,d^2\,B^2 + 108\,C^3\,A^3 - 1296\,d\,A^2\,C^2$$
$$+5184\,C\,A\,d^2 - 6912\,d^3))\hat{}(1/3) + 9\,A^2 - 24\,B + \Big(54\,\Big(A^3 - 4\,A\,B + 8\,C + \frac{1}{2}\,A^2\,\mathrm{sqrt}(6\,(8\,B^3 - 36\,A\,B\,C + 108\,d\,A^2 - 288\,d\,B$$
$$+108\,C^2 + 12\,\mathrm{sqrt}(-54\,A^3\,B\,C\,d + 240\,A\,B^2\,C\,d + 12\,B^3\,d\,A^2 - 3\,A^2\,B^2\,C^2 - 54\,A\,B\,C^3 - 432\,d^2\,A^2\,B + 18\,d\,A^2\,C^2$$
$$-432\,d\,B\,C^2 - 48\,B^4\,d + 12\,B^3\,C^2 + 81\,d^2\,A^4 + 384\,d^2\,B^2 + 12\,C^3\,A^3 + 81\,C^4 - 768\,d^3 + 576\,C\,A\,d^2))\hat{}(1/3) + 6\,(8\,B^3$$
$$-36\,A\,B\,C + 108\,d\,A^2 - 288\,d\,B + 108\,C^2 - 4\,\mathrm{sqrt}(59049 - 4\,B^6 + 36\,B^4\,A\,C - 144\,B^4\,d - 108\,A^2\,B^2\,C^2 + 864\,A\,B^2\,C\,d$$
$$-1728\,d^2\,B^2 + 108\,C^3\,A^3 - 1296\,d\,A^2\,C^2 + 5184\,C\,A\,d^2 - 6912\,d^3))\hat{}(1/3) + 9\,A^2 - 24\,B) - \frac{4}{3}\,B\,\mathrm{sqrt}(6\,(8\,B^3 - 36\,A\,B\,C$$
$$+108\,d\,A^2 - 288\,d\,B + 108\,C^2 + 12\,\mathrm{sqrt}(-54\,A^3\,B\,C\,d + 240\,A\,B^2\,C\,d + 12\,B^3\,d\,A^2 - 3\,A^2\,B^2\,C^2 - 54\,A\,B\,C^3 - 432\,d^2\,A^2\,B$$

$$+ 18\,d\,A^2 C^2 - 432\,d\,B\,C^2 - 48\,B^4 d + 12\,B^3 C^2 + 81\,d^2 A^4 + 384\,d^2 B^2 + 12\,C^3 A^3 + 81\,C^4 - 768\,d^3 + 576\,C\,A\,d^2))^\wedge(1/3) +$$
$$6\,(8\,B^3 - 36\,A\,B\,C + 108\,d\,A^2 - 288\,d\,B + 108\,C^2 - 4\,\mathrm{sqrt}(59049 - 4\,B^6 + 36\,B^4 A\,C - 144\,B^4 d - 108\,A^2 B^2 C^2$$
$$+ 864\,A\,B^2 C\,d - 1728\,d^2 B^2 + 108\,C^3 A^3 - 1296\,d\,A^2 C^2 + 5184\,C\,A\,d^2 - 6912\,d^3))^\wedge(1/3) + 9\,A^2 - 24\,B) - \frac{1}{27}(6\,(8\,B^3$$
$$- 36\,A\,B\,C + 108\,d\,A^2 - 288\,d\,B + 108\,C^2 + 12\,\mathrm{sqrt}(-54\,A^3 B\,C\,d + 240\,A\,B^2 C\,d + 12\,B^3 d\,A^2 - 3\,A^2 B^2 C^2 - 54\,A\,B\,C^3$$
$$- 432\,d^2 A^2 B + 18\,d\,A^2 C^2 - 432\,d\,B\,C^2 - 48\,B^4 d + 12\,B^3 C^2 + 81\,d^2 A^4 + 384\,d^2 B^2 + 12\,C^3 A^3 + 81\,C^4 - 768\,d^3$$
$$+ 576\,C\,A\,d^2))^\wedge(1/3) + 6\,(8\,B^3 - 36\,A\,B\,C + 108\,d\,A^2 - 288\,d\,B + 108\,C^2 - 4\,\mathrm{sqrt}(59049 - 4\,B^6 + 36\,B^4 A\,C - 144\,B^4 d$$
$$- 108\,A^2 B^2 C^2 + 864\,A\,B^2 C\,d - 1728\,d^2 B^2 + 108\,C^3 A^3 - 1296\,d\,A^2 C^2 + 5184\,C\,A\,d^2 - 6912\,d^3))^\wedge(1/3) + 9\,A^2 - 24\,B)^\wedge(3/2$$
$$)\Big)\Big)\Big/(\mathrm{sqrt}(6\,(8\,B^3 - 36\,A\,B\,C + 108\,d\,A^2 - 288\,d\,B + 108\,C^2 + 12\,\mathrm{sqrt}(-54\,A^3 B\,C\,d + 240\,A\,B^2 C\,d + 12\,B^3 d\,A^2$$
$$- 3\,A^2 B^2 C^2 - 54\,A\,B\,C^3 - 432\,d^2 A^2 B + 18\,d\,A^2 C^2 - 432\,d\,B\,C^2 - 48\,B^4 d + 12\,B^3 C^2 + 81\,d^2 A^4 + 384\,d^2 B^2$$
$$+ 12\,C^3 A^3 + 81\,C^4 - 768\,d^3 + 576\,C\,A\,d^2))^\wedge(1/3) + 6\,(8\,B^3 - 36\,A\,B\,C + 108\,d\,A^2 - 288\,d\,B + 108\,C^2 - 4\,\mathrm{sqrt}(59049 - 4\,B^6$$
$$+ 36\,B^4 A\,C - 144\,B^4 d - 108\,A^2 B^2 C^2 + 864\,A\,B^2 C\,d - 1728\,d^2 B^2 + 108\,C^3 A^3 - 1296\,d\,A^2 C^2 + 5184\,C\,A\,d^2 - 6912\,d^3)$$
$$)^\wedge(1/3) + 9\,A^2 - 24\,B))\Bigg) - \frac{1}{4}A$$

The other solutions are similar.

References:

Devaney, Robert L. An Introduction to Chaotic Dynamical Systems (2nd Edition). Redwood City, California : Addison-Wesley Publishing Co. Inc., 1989.

Jacobson, Nathan. Basic Algebra I (2nd Edition). New York : W H Freeman & Co., 1985.

Rotman, Joseph J. An Introduction to the Theory of Groups. Newton, Massachusetts : Allyn and Bacon, Inc., 1984

Biography

Fred Worth received his B.S. in Mathematics from Evangel College in Springfield, Missouri in 1982. He received his M.S. in Applied Mathematics in 1987 and his Ph.D. in Mathematics in 1991 from the University of Missouri-Rolla where his son is currently attending school. He has been teaching at Henderson State University since August 1991. He is a member of the Society for American Baseball Research, the Mathematical Association of America and the Association of Christians in the Mathematical Sciences.