A Markov Chain Competition Model

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Abstract

A birth and death chain for two or more species is examined analytically and numerically.

Description of the Model

Let any S be any finite list. Consider the chain where a random term of S replaces a random term of S. This was originally Fred Worth's idea where he simulated this manually on his TI-83 using list operations. This can be thought of as a finite population of several species that compete with each other for space. I will refer to the first selected term as the aggressor and the second selected term the victim.

This chain is similar to the Game of Life which was invented by John Conway in 1970. This game takes place in a two-dimensional matrix where each entry is either populated (alive) or unpopulated (dead). Here are the rules for the Game of Life:

1. For a space that is 'populated':

> Each cell with one or no neighbors dies, as if by loneliness.
> Each cell with four or more neighbors dies, as if by overpopulation.
> Each cell with two or three neighbors survives.

2. For a space that is 'empty' or 'unpopulated', each cell with three neighbors becomes populated.

The Game of Life unfolds in a two-dimensional matrix where proximity is important. However, the chain discussed in this paper takes place in a space that is small enough that an aggressor can instantly jump to any other location to claim his victim.

Here are some obvious observations about our model:

- This is a Markov chain since the probability of the next state depends only on the previous state.
- The absorbing states are where all but one species is extinct.
- The number of states is m^n where m is the number of species and n is the size of the list.

The original problem is difficult, so I will instead simplify by considering just two species and only keeping track of the number of species, not the position. Then the state space becomes $\{0,1,2,\ldots,n\}$ where n is the size of the list. Each state is the population size for the first species.

Here is an example of a list for $n = 7$ that was generated using the TI-83/4 program CHAIN. (A list of the programs appears in appendix.) Each column is a state and it took 33 steps to reach equilibrium. Note that the single individual 1 in the initial state eventually wiped out the species 0.

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Transition and Limiting Probabilities

It is left as an exercise for the reader to derive the following transition probabilities. (The proof depends on the aggressor and victim being chosen independently.)

$$p_{ij} = \begin{cases} i(n-i)/n^2 \text{ if } j = i \pm 1 \\ \left(i^2 + (n-i)^2\right)/n^2 \text{ if } j = i \\ 0 \text{ otherwise} \end{cases} \qquad \text{where } 0 \le i, j \le n$$

This is the probability of moving from state i to state j. Note that this is a birth and death chain where moving from state i to i+1 is a birth and moving from state i to i-1 is a death.

The following probability transition matrices were generated using TI-83 program PROBMAT. The rows and columns are numbered 0..n. For example, if $n = 3$, then the probability of going from 1 to 0 is 2/9, staying at 1 is 5/9, and going from 1 to 2 is 2/9.

$$(n=2) \quad \begin{bmatrix} 1 & 0 & 0 \\ .25 & .50 & .25 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(n=3) \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ .\overline{2} & .\overline{5} & .\overline{2} & 0 \\ 0 & .\overline{2} & .\overline{5} & .\overline{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(n=4) \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ .1875 & .625 & 0.1875 & 0 & 0 \\ 0 & .25 & .5 & .25 & 0 \\ 0 & 0 & 0.1875 & .625 & 0.1875 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(n=5) \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ .16 & .68 & .16 & 0 & 0 & 0 \\ 0 & .24 & .52 & .24 & 0 & 0 \\ 0 & 0 & .24 & .52 & .24 & 0 \\ 0 & 0 & 0 & .16 & .68 & .16 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The matrices for n odd are diagonally dominant. The matrices for n even are weakly diagonally dominant because the $p_{ii} = 0.5$ and $p_{i,i-1} = p_{i-1,i} = 0.25$ when $i = n/2$. The diagonally dominance can be proven starting with $(2i/n - 1)^2 \ge 0$ and then showing that $i^2 + (n - i)^2 \ge 2i(n - i)$.

Recall that the limiting probability matrix is the limit of P^n as *n* approaches infinity where $P = (p_{ij})$ is the probability transition matrix.

Numerical evidence obtained by quickly raising transition probability matrices to the 255th power using a TI-83 suggests that the probability of the species starting with i individuals winning is i/n. The following example shows the limiting probability for the case where $n = 3$.

$$\lim_{n\to\infty}\begin{bmatrix} 1 & 0 & 0 & 0 \\ .\overline{2} & .\overline{5} & .\overline{2} & 0 \\ 0 & .\overline{2} & .\overline{5} & .\overline{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ .\overline{6} & 0 & 0 & .\overline{3} \\ .\overline{3} & 0 & 0 & .\overline{6} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This shows that "the bigger army is more likely to win". I can prove this for any specific value of n using the eigenpairs of P, but I have not been able to prove it in general. Note that the probability that the single 1 wins is 1/7 for the numerical example given earlier in this paper.

Random Walk View

It is possible to use the transition probabilities to construct a random walk on the consecutive integers {0,1,2,…,n} by using the transitional probabilities for moving left, staying, and moving right, respectively.

$$\begin{cases} P[i \text{ to } i-1] = i(n-i)/n^2 \\ P[\text{stay at } i] = \left(i^2 + (n-i)^2\right)/n^2 \\ P[i \text{ to } i+1] = i(n-i)/n^2 \end{cases} \quad \text{where } i = 0,1\ldots n$$

Let T_i ($0 \le i \le n$) be the time for the walk to reach the end (0 or 1). Then

$$T_0 = T_n = 0$$

$$T_i = 1 + \begin{cases} T_{i\pm1} \text{ wp } i(n-1)/n^2 \\ T_i \text{ wp } \left(i^2 + (n-i)^2\right)/n^2 \end{cases} \quad \text{where } 0 < i < n$$

The following system of expectations follow:

$$\begin{cases} E[T_i] = 1 + \dfrac{1}{n^2}\left(i(n-i)E[T_{i-1}] + \left(i^2 + (n-i)^2\right)E[T_i] + i(n-i)E[T_{i+1}]\right) \\ E[T_0] = E[T_n] = 0 \text{ where } 0 < i < n \end{cases}$$

This simplifies to the following tridiagonal system:

$$-E[T_{i-1}] + 2E[T_i] - E[T_{i+1}] = \frac{n^2}{i(n-i)} \quad \text{where } 0 < i < n$$

The following table gives the solution for the expected time to end for *n* up to seven.

n, the number of elements in the space	The expected values.
2	$E[T_1]=2$
3	$E[T_1]=4.5$
4	$E[T_1]=7.333$, $E[T_2]=9.333$
5	$E[T_1]=10.417$, $E[T_2]=14.583$
6	$E[T_1]=13.7$, $E[T_2]=20.2$, $E[T_3]=22.2$
7	$E[T_1]=17.15$, $E[T_2]=26.133$, $E[T_3]=30.217$

By the symmetry $T_i = T_{n-i}$, so we only have to find half of the expected values of T_i. For example, when $n = 3$, $E[T_2] = 4.5$. The program ETIMES was used to compute these expected times. The program SIM was used to simulate the time to end and confidence intervals were computed to support the above results.

The distribution for the time to end T_1 is Geometric(1/2) when $n = 2$ can be easily proven using the transition probabilities. I determined that the distribution for T_1 is Geometric (2/9) when $n = 3$ numerically. I do not know if the distribution for T_i for $n \geq 4$ has a name, but it can be determined numerically by using $P[T_i \leq k] = \left(P^k\right)_{i0} + \left(P^k\right)_{in}$. $(P^k)_{ij}$ means the ith row and the jth column of the matrix P^k. That is, add the ends of i^{th} row of the k^{th} power of P to get the cumulative distribution function for T_i.

I will briefly consider the time to required to reach a particular end. Let $L_i = T_i|$reach 0 be the time to reach the left end, and $R_i = T_i|$reach n be the time to reach the right end. Then

$$E[T_i] = E[L_i]P[\text{reach } 0] + E[R_i]P[\text{reach } n] = E[L_i]\left(\frac{n-i}{n}\right) + E[L_{n-i}]\left(\frac{i}{n}\right),\ 0 < i < n$$

since $R_i = L_{n-i}$ by symmetry. The nonzero entries of this sparse coefficient matrix when solving for $E[L_i]$ look like an X. The interested reader can solve this linear system as was done earlier for $E[T_i]$.

A Brief Look at Multiple Species

Suppose that we allow n $n \geq 2$ species to compete. For $i = 1..n$, we have the following a priori estimates:

- P[ith species wins] = i/n
- The time for one species to win is at least $\max(T_{n_i}, i = 1..n)$.

These two facts follow by partitioning the species into two groups and using the previous results for 2 species.

Ideas for Further Work

- Determine if the distribution of T_i has a name where $i \geq 4$.
- Develop a continuous version.
- Develop a 2-dimensional version like Conway's Game of Life.
- Investigate multiple species.
- Supply a proof of the limiting probabilities for all n.

References

- *Conway's Game of Life* on Wikipedia (http://en.wikipedia.org/wiki/Conway's_Game_of_Life)
- *Introduction to Stochastic Processes* by Paul Hoel, Sidney Port, and Charles Stone ©1972 by Houghton Mifflin

Appendix of TI-83/4 Programs

CHAIN

```
Input "SIZE=",N
Input "P(1)=",P
N→dim(L1)
seq(rand<P,J,1,N)→L1
1→J
List▶matr(L1,[A])
Disp L1
Repeat S=0 or S=N
J+1→J
L1(randInt(1,N))→L1(randInt(1,N))
{N,J}→dim([A])
For(I,1,N)
L1(I)→[A](I,J)
End
sum(L1)→S
Disp L1
End
Disp J
```

PROBMAT

```
Prompt N
N+1
{Ans,Ans}→dim([A])
Fill(0,[A])
For(I,1,N-1)
I(N-I)→[A](I+1,I)
Ans→[A](I+1,I+2)
I²+(N-I)²→[A](I+1,I+1)
End
N²→[A](1,1)
Ans→[A](N+1,N+1)
Disp [A]
```

ETIMES

```
Prompt N
identity(N+1)→[A]
{N+1,1}→dim([B])
0→[B](1,1)
0→[B](N+1,1)
For(I,1,N-1)
⁻.5→[A](I+1,I)
Ans→[A](I+1,I+2)
N²/2/I/(N-I)→[B](I+1,1)
End
augment([A],[B])
rref(Ans)→[A]
N+1→dim(L1)
For(I,1,N+1)
[A](I,N+2)→L1(I)
End
Pause L1
```

SIM

```
Prompt I,N,M
M→dim(L1)
M→dim(L2)
For(K,1,M)
0→T
I→J
Repeat J=0 or J=N
T+1→T
J(N-J)/N²→P
rand→R
If R<P
Then
J-1→J
Else
If R>1-P
J+1→J
End
End
J→L2(K)
T→L1(K)
End
```

Biography

Michael Lloyd received his B.S in Chemical Engineering in 1984 and accepted a position at Henderson State University in 1993 shortly after earning his Ph.D. in Mathematics from Kansas State University. He has presented papers at meetings of the Academy of Economics and Finance, the American Mathematical Society, the Arkansas Conference on Teaching, the Mathematical Association of America, and the Southwest Arkansas Council of Teachers of Mathematics. He has also been an AP statistics consultant since 2002.