ON PERIODIC POINTS OF MAPS OF TREES AND THE EXPANSIVE PROPERTY

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Abstract

In this paper, we will consider a couple of preliminary results regarding fixed points of certain continua. These results are used to lead to the main result, namely that if f is a mapping of a tree, T, into itself and if $T_0 = \bigcap_{n=1}^{\infty} f^n(T)$ contains at least two points then there is a positive integer, k, such that f^k has two fixed points. In particular, k is not greater than the number of end points of T. This result leads to an elementary proof of a result due to Kato, that shift homeomorphisms of the inverse limit of a tree with a single bonding map are not expansive.

Terminology and Introduction

F or a function, f, and a positive integer, n, we define the following: $f^1 = f$, $f^n = f(f^{n-1})$, $f^{-n} = (f^{-1})^n$. Also, f^0 will denote the identity function. A homeomorphism, f, of a metric space, X, onto itself is called **expansive** if there is a positive number, ε, such that if x and y are distinct points of X, then there exists an integer, n = n(x,y), such that $d(f^n(x), f^n(y)) > \varepsilon$. It should be noted that n may be negative. The positive number, ε, is called an **expansive constant**. This concept, using the term **unstable homeomorphism**, was first defined in 1950 by Utz [4]. In 1955, Gottschalk and Hedlund changed the name to **expansive homeomorphism** [1]. The term "unstable" is probably more accurate, since two points could be moved farther apart and then come back together again. "Expansive" seems to imply a more regular process of increasing distances. In fact, "expansive" is used in some areas of topology to mean exactly that.

By a **mapping**, we mean a continuous function. By a **continuum**, we mean a compact, connected metric space. A **tree** is a finite, connected union of arcs, containing no simple closed curve.

Let X be a topological space. Let f be a mapping of X to itself. Let M be the subset of the product space $X \times X \times \dots$ with the property that $(x_1, x_2, \dots) \in M$ if and only if $f(x_{i+1}) = x_i$, for i = 1, 2, ... Then M is called the **inverse limit** of X with the single bonding map, f, and is denoted {X, f}. If X is a metric space with metric d, we define a metric, ρ, on M by $\rho((x_1, x_2, \dots), (y_1, y_2, \dots)) = \sum_{n \geq 1} d(x_n, y_n)/2^n$.

There is a natural homeomorphism of M onto itself defined as follows: $h(x_1, x_2, \dots) = (f(x_1), x_1, x_2, \dots)$. This homeomorphism is called the **shift homeomorphism**. Let f be a map of X to itself. The **orbit** of x under f is defined by O(x) = $\cup_{n=-\infty}^{\infty} f^n(x)$. A **suborbit** of x under f is any doubly infinite sequence { . . . , x_{-2}, x_{-1}, x_0, x_1, x_2, . . . } such that $x_0 = x$ and $f(x_{i+1}) = x_i$, for all integers i.

The mapping f is **expansive** on X, with **expansive constant** $\varepsilon > 0$, if x and y in X with $x \neq y$ implies, for each suborbit, O(x) of x and O(y) of y, there exists an integer n such that $d(x_n, y_n) > \varepsilon$.

Preliminary Results

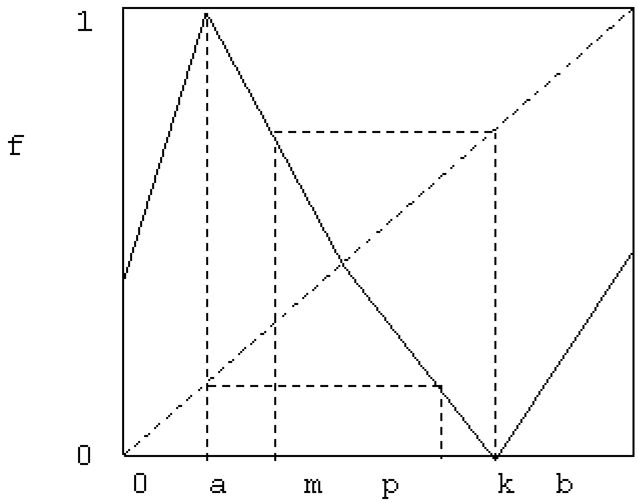
The proof of this first result is a simple exercise.

Lemma 1. Let f be a map of a metric space X onto itself. The shift homeomorphism on M = {X, f} is expansive if and only if f is expansive.

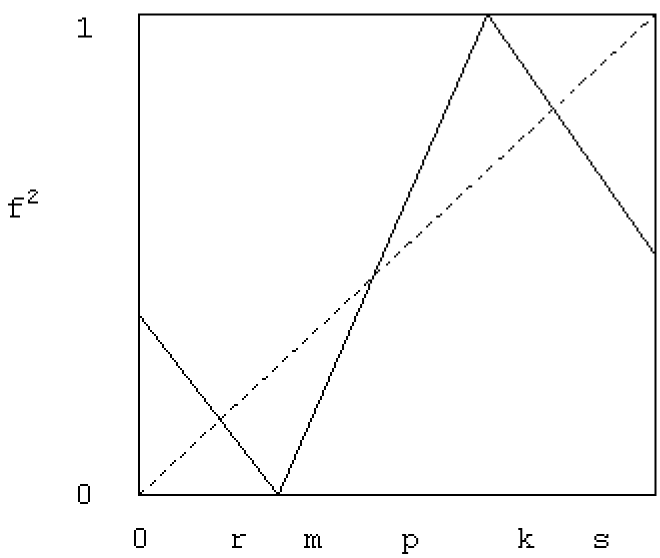
Lemma 2. If f is a map of [0, 1] onto itself, then f^2 has at least two fixed points.

Proof. It is trivial to show that f has a fixed point. If f(0) = 0 or f(1) = 1, then f must have two fixed points. We will consider only the case where f has only one fixed point.

Let p be the fixed point of f. There exist points, a and b, such that $f(a) = 1$ and $f(b) = 0$. It is not hard to show that we must have $a < p < b$. Additionally, there must be points, k and m, with $a < m < p < k < b$, such that $f(m) = b$ and $f(k) = a$.



Then, $f^2(k) = f(f(k)) = f(a) = 1$ and $f^2(m) = f(f(m)) = f(b) = 0$. We must, then, by the Intermediate Value Theorem have points *r* and *s*, with 0 £ *r* < *m* < *p* < *k* < *s* £ 1 such that $f^2(r) = r$ and $f^2(s) = s$.



QED

The next result comes with an interesting historical note. It was proved by R. K. Williams of Southern Methodist University. At the same time that Williams was doing some of this research, the author of this paper was a student in Williams' Calculus 2 class. The proof will not be given.

Lemma 3. Let f be a map of a compact metric space onto itself. If the set of fixed points of f is infinite then f is not expansive. Further, if for some positive integer, n, the set of points of period n is infinite, f is not expansive. [4, 5]

Theorem 1. Let f be a uniformly continuous map. Then f^n is expansive if and only if f is expansive for any nonzero integer n. [4, 5]

Lemma 4. If g is a map of [0, 1] onto itself, then g is not expansive. [5]

Proof. If g has infinitely many fixed points we are done. Similarly for g^2**.** Let f = g^2**.**

Let p and q be fixed points of f, with p < q, such that no point between p and q is a fixed point of f. Let e > 0 be given.

We will assume that f(x) > x for all x Î (p, q). The case where f(x) < x is similar.

Suppose, first, that f(x) is strictly increasing on (p, q). Let a Î (p, p + e). Let a_0 = a. For n > 0, let $a_n = f^n(a)$. For n < 0, let a_n be a point in (p, a_{n+1}) such that $f(a_n) = a_{n+1}$. Let N be the least positive integer such that a_N Î (q - e, q). Since f is uniformly continuous, we can choose d > 0 such that $|f^i(x) - f^i(y)| < e$ for all i Î {1, 2, . . . , N} whenever |x - y| < d. Choose b Î (p, p + e) Ç (a - d, a + d). Let b_0 = b. For n > 0, let $b_n = f^n(b)$. For n < 0, let b_n be a point in (p, b_{n+1}) such that $f(b_n) = b_{n+1}$. For all n, $|b_n - a_n| < e$. Thus f is not expansive.

Suppose f achieves a local maximum, say f(k), on (p, q). For n < 0, let k_n be a point in (p, k_{n+1}) such that $f(k_n) = k_{n+1}$. Let N be the greatest integer such that $|k_N - p| < e$. Since f is uniformly continuous, we can choose d > 0 such that $|f^i(x) - f^i(y)| < e$ for all i Î {1, 2, . . . , -N} whenever |x - y| < d. Choose a Î (p, k_N) Ç (k_N - d , k_N + d). Choose b Î (k_N , p + e) Ç (k_N - d , k_N + d). If $f^{-N}(a) = f^{-N}(b)$, then let c = a and d = b. If $f^{-N}(a) < f^{-N}(b)$, then choose c Î (a, k_N) such that $f^{-N}(c) = f^{-N}(b)$. Let d = b. If $f^{-N}(a) > f^{-N}(b)$, then choose d Î (k_N, b) such that $f^{-N}(a) = f^{-N}(d)$. Let c = a. Let c_0 = c. For n > 0, let $c_n = f^n(c)$. For n < 0, let c_n be a point in (p, c_{n+1}) such that $f(c_n) = c_{n+1}$. Let d_0 = d. For n > 0, let $d_n = f^n(d)$. For n < 0, let d_n be a point in (p, d_{n+1}) such that $f(d_n) = d_{n+1}$. For all n, $|c_n - d_n| < e$. Thus, f is not expansive. Thus, g is not expansive.

QED

Theorem 2 follows from Lemmas 1 and 3. Theorem 2 first appeared, without proof, in a paper by Jakobsen and Utz in 1960.

Theorem 2. Let A be an arc and M = {A, f}, where f is a map of A onto itself. Then the shift homeomorphism on M is not expansive. [2]

Lemma 5 is an easy exercise.

Lemma 5. Let f be a map of a continuum, X, onto itself. Let n be a positive integer. Then the shift homeomorphism on {X, f} is expansive if and only if the shift homeomorphism on $\{X, f^n\}$ is expansive.

Main Result

We will adopt the following convention. Let T be a tree and let A and B be points of T. Let [AB] be the sub-arc of T with endpoints A and B. There is a homeomorphism, g, of [AB] onto [0, 1], with g(A) = 0 and g(B) = 1. For x and y in [AB], we say $x < y$ if $g(x) < g(y)$. Let [AB] be an arc in T. Then [AB] is said to satisfy **Condition 1** if there exist a branch point, x, and a point, y, in [AB] such that f(y) = x, $y < x$ and f(yB] does not intersect [AB]. We say [AB] satisfies **Condition 2** if there exist a branch point, x, and distinct points, y and z, in [AB] such that $y < x < z$, f(y) = f(z) = x and f(yz) does not intersect [AB].

Theorem 5 was proved earlier if the tree is an arc. Therefore, in the proofs we will assume the tree has at least one branch point. Theorem 3 is a well known result. We include the proof since the method of proof will be helpful in proving Theorem 5 below. The essential idea of the proof is a "dog chases rabbit" argument.

Theorem 3. Trees have the fixed point property.

Proof. Let T be a tree and f a mapping of T into itself. Let O be a branch point of T. If no such point exists, then T is an arc. Suppose $f(O) = P \neq O$. Let A_1 be an endpoint of T such that P is in $[OA_1]$. If there are no branch points in $(OA_1]$ then f restricted to $[OA_1]$ has a fixed point.

Suppose f does not have a fixed point in $[OA_1]$. Then either Condition 1 or Condition 2 must hold on $[OA_1]$. In either case, we have a branch point, B_1, in $(OA_1]$ and a point, y, of $[OA_1]$ such that $y < B_1$, $f(y) = B_1$ and $f(B_1)$ is not in $[OA_1]$. Choose an end point, A_2, of T, such that $f(B_1)$ is in $[B_1A_2]$.

Inductively, suppose f does not have a fixed point in $[B_iA_{i+1}]$. Then either Condition 1 or Condition 2 must hold. In either case, we have a branch point, B_{i+1}, in $(B_iA_{i+1}]$ and a point, y, of $[B_iA_{i+1}]$ such that $y < B_{i+1}$, $f(y) = B_{i+1}$ and $f(B_{i+1})$ is not in $[B_iA_{i+1}]$. Choose an end point, A_{i+2}, of T, such that $f(B_{i+1})$ is in $[B_{i+1}A_{i+2}]$.

Since the number of branch points is finite, there exists a positive integer, k, such that no branch point is in $(B_kA_{k+1}]$. Then f has a fixed point in $[B_k, A_{k+1}]$. **QED**

Theorem 4. Let f be a map of a tree, T, onto itself. Then, if N is the number of end points of T, there exists a positive integer k £ N such that for some non-degenerate sub-arc [OA] of T, $f^k[OA]$ Ê [OA].

Proof. Let O be a fixed point for f and let $A_1, . . . , A_N$ be the end points of T. Let $I_i = [OA_i]$. Suppose O ¹ A_1. Let $R_0 = A_1$. There exists a point, P_1, of T such that $f(P_1) = R_0$. P_1 is in some I_j. Let $R_1 = A_j$. Thus, $f[OR_1]$ Ê $[OR_0]$.

Inductively, there exists a point, P_{i+1}, of T such that $f(P_{i+1}) = R_i$. If P_{i+1} is in I_j then let $R_{i+1} = A_j$. Thus, $f[OR_{i+1}]$ Ê $[OR_i]$. Also, $f^r[OR_{i+1}]$ Ê $[OR_{i-r+1}]$ for $r = 1, 2, . . . , i+1$.

There exist k and s such that 0 < k - s £ N and $R_k = R_s$. Therefore, $f^{k-s}[OR_k]$ Ê $[OR_s] = [OR_k]$.

QED

We can now prove our main result.

Theorem 5. Let f be a mapping of a tree, T, into itself. If $T_0 = \bigcap_{n=1}^{\infty} f^n(T)$ contains at least two points then there is a positive integer, k, such that f^k has two fixed points. In particular, k is not greater than the number of end points of T.

Proof. If T_0 contains at least two points, then T_0 is a tree and $f(T_0) = T_0$. Thus it suffices to consider the case where f is a surjection.

By Theorem 3, we know f has a fixed point, O. Let N denote the number of end points of T. By Theorem 4, we know there exists an arc, [OA], and a positive integer, k £ N, such that f^k[OA] contains [OA]. Let $g = f^k$.

Let B be the set of branch points of T. Suppose g fails to have a second fixed point in [OA]. Since g[OA] contains [OA], either Condition 1 or Condition 2 holds. Let B_1 = x from Condition 1 or Condition 2. Choose an end point, A_1, of T such that g(x) is in [OA_1].

Following the proof of Theorem 3, we obtain a branch point, B_i and an end point, A_i, such that (B_iA_i] contains no branch point of T and $g(B_i)$ is in [B_iA_i]. Thus, g has a fixed point in [B_iA_i] and this point is not O. **QED**

A Corollary

We give the following lemma without proof.

Lemma 6. (Intermediate value theorem for trees) Let f be a mapping of a tree, T, into itself. Let [AB] be an arc in T such that f[AB] contains [AB]. Suppose x and y are members of [AB] with x < y and f(x) and f(y) are in [AB] with f(x) ¹ f(y). Then for any member, z¢, in (f(x),f(y)), there exists z in (x,y) such that f(z) = z¢.

In [3], Kato proved that the shift homeomorphism on M is not expansive. Let f be a map of a tree, T, to itself. If $T_0 = \cap_{n=1}^{\infty} f^n(T)$ is a singleton, then the inverse limit, (T, f) is degenerate. If T_0 is not a singleton, then it is a non-degenerate subtree of T such that $f(T_0) = T_0$. If the shift homeomorphism on (T_0, f) is not expansive then the shift homeomorphism on (T, f) will not be expansive. Thus, the following theorem will establish Kato's result.

Theorem 6. Let T be a tree and let f be a map of T onto itself. Let M = {T, f}. Then the shift homeomorphism on M is not expansive.

The proof of this result is given in [6].

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Biographical Sketch

Fred Worth received his B.S. in Mathematics from Evangel College in Springfield, Missouri in 1982. He received his M.S. in Applied Mathematics in 1987 and his Ph.D. in Mathematics in 1991 from the University of Missouri at Rolla. He has been teaching at Henderson State University since August 1991.